Uncertainty Principle

Note Title 9/13/2010

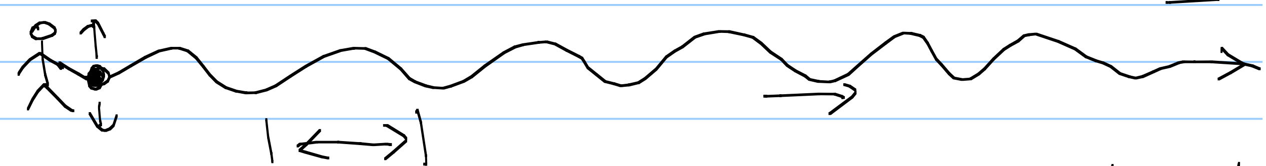
$$\boxed{\sigma_x \sigma_p \geq \frac{\hbar}{2}}$$

[or $\Delta x \Delta p \geq \frac{\hbar}{2}$ in another notation]

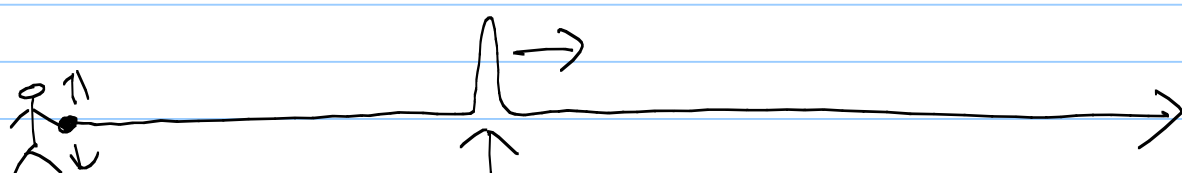
* Meaning : well-defined position
  $\Rightarrow$ ill-defined momentum
  vice versa.

* well-defined momentum is equivalent to well-defined wave length according to the "de Broglie formula" $p = \frac{h}{\lambda} = \frac{2\pi\hbar}{\lambda} = \hbar k$

[$k = \frac{2\pi}{\lambda}$: definition]

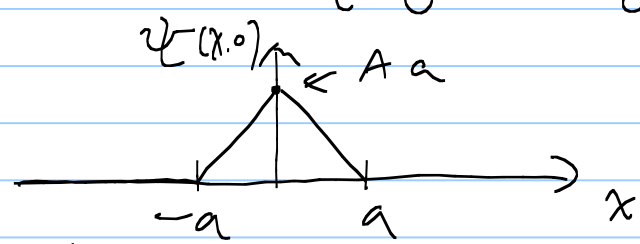
① 

λ : well-defined, position : ill-defined

② 

x : well-defined
(position)
λ : ill-defined

Ex. $\Psi(x, 0) = \begin{cases} A(a - |x|) & \text{for } -a \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$



what are $\langle x \rangle$, $\langle p \rangle$, $\langle x^2 \rangle$, $\langle p^2 \rangle$, $\langle \sigma_x \rangle$, $\langle \sigma_p \rangle$ and check the uncertainty principle

(a) Normalize

$$1 = \int_{-\infty}^{\infty} |\Psi|^2 dx = \int_{-a}^{a} [A(a-|x|)]^2 dx$$

$$= 2A^2 \int_0^a (a-x)^2 dx = 2A^2 \int_0^a (a^2 - 2ax + x^2)\, dx$$

$$= 2A^2 \left[a^2 x - a x^2 + \frac{x^3}{3} \right]_0^a$$

$$= 2A^2 \left[a^3 - a^3 + \frac{a^3}{3} \right] = \frac{2a^3}{3} A^2$$

$$\Rightarrow \quad A = \left[\frac{3}{2a^3} \right]^{\frac{1}{2}}$$

(b) $$\langle x \rangle = A^2 \int_{-a}^{a} x (a-|x|)^2 dx$$

$$= 0, \text{ (odd integrand)}$$

(c) $$\langle p \rangle = A^2 \int_{-a}^{a} (a-|x|) \frac{\hbar}{i} \frac{\partial}{\partial x} (a-|x|)\, dx$$

$$= -i\hbar A^2 \left[\int_0^a (a-x) \frac{\partial}{\partial x} (a-x)\, dx + \int_{-a}^{0} (a+x) \frac{\partial}{\partial x} (a+x)\, dx \right]$$

$$= -i\hbar A^2 \left[-\int_0^a (a-x)\, dx + \int_{-a}^{0} (a+x)\, dx \right]$$

$$= -i\hbar A^2 \left[-\left(ax - \frac{x^2}{2} \right) \Big|_0^a + \left(ax + \frac{x^2}{2} \right) \Big|_{-a}^{0} \right]$$

$$= -i\hbar A^2 \left[-\left(a^2 - \frac{a^2}{2} \right) - \left(-a^2 + \frac{a^2}{2} \right) \right]$$

$$= 0$$

(d) $\langle x^2 \rangle = A^2 \int_{-a}^{a} x^2 (a-|x|)^2 dx$

$= 2A^2 \int_0^a x^2 (a-x)^2 dx$

$= 2A^2 \int_0^a (a^2x^2 - 2ax^3 + x^4)\, dx$

$= 2A^2 \left[\frac{a^2x^3}{3} - \frac{ax^4}{2} + \frac{x^5}{5} \right]_0^a$

$= 2A^2 \left[\frac{a^5}{3} - \frac{a^5}{2} + \frac{a^5}{5} \right]$

$= 2A^2 \frac{10-15+6}{30} a^5 = \frac{1}{15} A^2 a^5$

$= \frac{1}{15} \cdot \frac{3}{2a^3} \cdot a^5 = \frac{a^2}{10}$

(e) $\langle p^2 \rangle = A^2 \int_{-a}^{a} (a-|x|)(-\hbar^2) \frac{\partial^2}{\partial x^2}(a-|x|)\, dx$

$= -A^2\hbar^2 \left[\int_0^a (a-x) \frac{\partial^2}{\partial x^2}(a-x)\, dx + \int_{-a}^{0} (a+x) \frac{\partial^2}{\partial x^2}(a+x)\, dx \right]$

[naive way]

$= -A^2\hbar^2 \left[\int_0^a (a-x)\, 0\, dx + \int_{-a}^{0} (a+x)\, 0\, dx \right]$

$= 0$

$\hookrightarrow$ wrong!

Correct way: $\frac{\partial}{\partial x}|x| = \begin{cases} 1 & \text{for } x>0 \\ -1 & \text{for } x<0 \end{cases}$

If we introduce the step function (see prob. 2.24)

$$\theta(x) \equiv \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x < 0. \end{cases}$$

with this $\frac{d}{dx}|x| = 2\theta(x) - 1$

From Prob. 2.24, $\frac{d\theta(x)}{dx} = \delta(x)$ ← delta function

, with $\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0)$.

We will learn more about delta function later on.

So with these

$$\frac{d^2}{dx^2}|x| = \frac{d}{dx}(2\theta(x) - 1) = 2\frac{d\theta(x)}{dx} = 2\delta(x)$$

Then $\langle p^2 \rangle = -\hbar^2 A^2 \int_{-a}^{a} (a - |x|) \frac{d^2}{dx^2}(a - |x|)\, dx$

$$= \hbar^2 A^2 \int_{-a}^{a} (a - |x|)\, 2\delta(x)\, dx$$

$$= \hbar^2 A^2 \cdot 2(a - 0)$$

$$= 2\hbar^2 A^2 a = 2\hbar^2 \frac{3}{2a^3} = \frac{3}{a^3}\hbar^2$$

* **Note:** Rigorously speaking, we should have used step functions at $x = a$ & $-a$.
Then we would have had delta functions $\delta(x \pm a)$ in the integrand.
But this would not change the final answer because $(a - |\pm a|) = 0$.

(f) $\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{a^2}{10}} = \frac{a}{\sqrt{10}}$

$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{3}{a^2}\hbar^2} = \sqrt{3}\,\frac{\hbar}{a}$

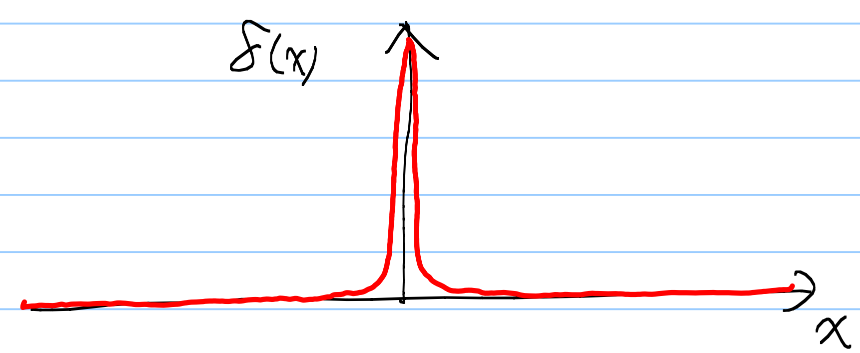
$\Rightarrow \sigma_x \cdot \sigma_p = \frac{a}{\sqrt{10}}\,\frac{\sqrt{3}}{a}\hbar = \sqrt{\frac{3}{10}}\,\hbar$

$> \frac{1}{2}\hbar$

$\therefore$ Uncertainty principle holds

* Let's do a little more discussion on the delta function: (from Griffiths 2.5.2)

$$\delta(x) \equiv \begin{cases} 0, & \text{for } x \neq 0 \\ \infty, & \text{for } x = 0 \end{cases}, \quad \int_{-\infty}^{\infty} \delta(x)\,dx = 1$$



$$f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)\int_{-\infty}^{\infty} \delta(x-a)\,dx$$

$$= f(a)$$

Also from Griffiths Eq [2.142]

$$\delta(cx) = \frac{1}{|c|}\,\delta(x) \Rightarrow \delta(-x) = \delta(x)$$